

Identifiers

DOI 10.46298/jtcam.15729

HAL hal-05079917v2

History

Received May 23, 2025

Accepted Jan 9, 2026

Published May 10, 2026

Associate Editor

Djimedou KONDO

Reviewers

Anonymous

Anonymous

Open Review

HAL hal-05528361

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On the intrinsic approaches in mechanics

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This paper aims to illustrate the advantages of the so-called intrinsic or coordinate-free or geometric approaches in mechanics and initiate a discussion on how these methods contrast with computational mechanics. Due to recent developments of pure computational approaches using Artificial Intelligence that could finally suggest that equations and formal developments are not necessarily so important in mechanics, it could obviously lead to philosophical and epistemological issues. Our main point of view can be summarized by the well-known sentence of René Thom: "To predict is not to explain". Even if we consider that stressing these questions in the community of mechanics should be fundamental and essential, it will not however be the theme of the present work. Instead, we will maintain a technical focus in order to defend the idea that intrinsic approaches are actually complementary to computational approaches in mechanics and that, even if these approaches may involve more abstract objects and less usual mathematical concepts, they can lead in fact to practically more efficient developments in applications. This claim supports the unity of mechanics and the non-sense to hermetically separate theoretical, computational and applied aspects in mechanics.

Keywords: coordinate-free mechanics, differential geometry

1 Introduction

Let us start by an example and let us consider a mechanical system constituted by five rigid bodies moving in the space and subjected by a family of kinematic constraints (joints between the bodies) so that the number of degrees-of-freedom is 21. Suppose that some elements of the system are not well known and that some parametric investigations will be necessary regarding the inertia operators of some of the five bodies. Suppose also that some external given actions are not conservative and that internal actions (joint interactions) are not perfectly known. How can we investigate this problem? The first remark is that it seems impossible to write by hand the 21 independent second order differential equations of the dynamics by Newton-Euler equations ($5 \times 6 = 30$ scalar equations + actions-reactions principle + constitutive laws of the joints) or equivalently the 21 Euler-Lagrange equations. For these last equations, the expression of the kinetic energy will be too cumbersome and unusable by mixing all the inertia of the different bodies and the loss of a potential energy will remove one of the greatest advantages consisting in concentrating the dynamics in a unique scalar function (the Lagrangian of the system). Since the system is not necessarily a usual robot, the machinery developed in robotics to parametrize the system cannot be directly used.

The approach usually taught (Newton-Euler or Lagrange as well) consists first in introducing a lot of coordinate systems in the (affine) space \mathcal{E} to describe the geometry of the mechanical system (positions of some points of \mathcal{E} and orientations of some bases of the vector space E associated to the affine space \mathcal{E}). Secondly, many derivative computations are performed to build the quantities that are introduced in the equations translating the principles of mechanics and leading to very expanded expressions. The additional mechanical features of the system that have to be taken account like the constitutive laws, are then translated into very expanded expressions due to the initial coordinates representations and calculations. Moreover, "models" of mechanical

systems are often presented as equations like

$$M(q)\ddot{q} + N(q, \dot{q})\dot{q} + G(q) + F(q, \dot{q}) = 0 \quad (1)$$

with $M(q)$, the mass matrix, $N(q, \dot{q})$, the matrix containing Coriolis and centrifugal forces, $G(q)$, the vector of positional forces and $F(q, \dot{q})$, a vector describing for example damping forces. A similar expression is the basic relation coming from Finite Element methods to investigate the dynamics of continuous systems, such as nonlinear mode computation. Such a formal expression is however only numerically calculated at each step (of the time interval or of the loading path) before solving it for the next step thanks to a linear development. In fact, these objects $M(q)$, $N(q, \dot{q})$, $G(q)$, and $F(q, \dot{q})$ are not explicitly calculated nor used as functions of the variables q . In fact, such an explicit expression is rather used in the dual problem of the control of the mechanical system when the trajectory $t \mapsto q(t)$ is prescribed and when motor actions in the links must be calculated to follow as well as possible the prescribed trajectory. Moreover, for such problems, the sensitivity of the system to some parameters is a fundamental question that leads for example to compute derivatives of $M(q)$, $N(q, \dot{q})$... with respect to these parameters. Doing so explicitly necessitates to have the most compact yet explicit expressions of these quantities. Unfortunately, the usual approaches do not provide such expressions since they carry the artificial complexity induced by coordinate systems used to represent any quantity by a family of real numbers. Roughly speaking, a coordinate-free approach proposes to report the use of coordinate systems at the end of the process and not at the beginning and to process all derivative calculations (kinematics, dynamics, sensibility, etc) before introducing coordinates.

Of course, the first contribution of intrinsic approaches is to identify the basic concepts linked to the investigated problems and to focus on the properties of these concepts without being diverted by useless computational developments that often could hidden the key features of the investigated problem. A second direct contribution (and often the most remarkable) is to solve a question that pure computational approaches could definitively not deal with. The third contribution most rarely emphasized is the practical aspects of the intrinsic approaches and the aid they can bring to computational methods. To illustrate our view, we will consider two very different kinds of applications of geometric concepts in mechanics. The first one deals with calculations on Lie groups whereas the second one deals with calculations in vector bundles. We will give (recent and “old”) results provided by such approaches and calculations in such spaces motivated by multi-body or continuous system dynamics for the Lie group approaches and by granular mechanics for the vector bundle concept.

2 Lie group approaches

2.1 An adapted coordinate-free approach

We first highlight the main elements of a coordinate-free approach by using calculations on Lie groups. These elements are completely developed in several publications but synthesized in the book (Chevallier and Lerbet 2018).

1. The set $\mathbb{D}(\mathcal{E})$ of affine Euclidean displacements in the 3-dimension affine space \mathcal{E} is a 6-dimensional Lie group $\mathbb{D}(\mathcal{E})$. Its Lie algebra $\mathcal{D}(\mathcal{E})$ is isomorphic to (and identified with) the set of skew-symmetric fields on \mathcal{E} (with values on E , the usual 3-dimension vector space associated to \mathcal{E}). On $\mathcal{D}(\mathcal{E})$, there is a $(3, 3)$ -bilinear symmetric form invariant by the adjoint action of the Lie group denoted Ad of $\mathbb{D}(\mathcal{E})$. It is the Klein form $[\cdot | \cdot]$ that allows to intrinsically identify the dual $\mathcal{D}(\mathcal{E})^*$ (describing the force fields acting on a rigid body) with $\mathcal{D}(\mathcal{E})$ describing the velocity fields of a rigid body (see hereafter). The set $\mathfrak{T}(\mathcal{E})$ of translations is an ideal of $\mathcal{D}(\mathcal{E})$ and is especially involved in the analysis of observation frames.
2. The so-called free and transitive action, denoted \bullet , of the 6-dimensional Lie group $\mathbb{D}(\mathcal{E})$ on the configuration space \mathbb{S}_S of any rigid body S allows us to endow this space to a differential manifold equipped with the structure of homogeneous space. In fact, this structure is exactly equivalent to the rigidity hypothesis for S . This mathematical structure also allows to develop a complete but explicit coordinate-free kinematics for the body S which is entirely described by the Maurer-Cartan form $\theta : T\mathbb{S}_S \rightarrow \mathcal{D}(\mathcal{E})$. Note that on a Lie group there are two Maurer-Cartan

- forms (a left one and a right one) whereas on a homogeneous space, we lose the symmetry of a Lie group structure and there is only one Maurer-Cartan form.
3. The additional hypothesis needed to deal with kinetics and dynamics is related to a Riemannian structure on \mathbb{S}_S that is equivalent to a given field $s \in \mathbb{S}_S \mapsto H_s \in \mathcal{D}(\mathcal{E})$ of linear operators of the Lie algebra $\mathcal{D}(\mathcal{E})$ of $\mathbb{D}(\mathcal{E})$ with some well-defined algebraic properties like symmetry with respect to the Klein form $[\cdot | \cdot]$. This linear operator translates the way to pass from the kinematics object namely the velocity field $V(t) = \theta(\dot{s}(t))$ to the kinetic momentum $H_{s(t)}(V(t))$ when the motion of S is $t \mapsto s(t)$ plotted on \mathbb{S}_S .
 4. The three previous items allow to deal with kinematics and dynamics of a single rigid body provided we give the appropriate statements of the Newton-Euler laws namely the principles of mechanics! In order to extend this full intrinsic calculation from a single rigid body S to a rigid bodies system $\Sigma = \{S_i, i = 1, \dots, n\}$, natural hypotheses are required. Preserving the Lie group approach leads to suppose that the links between any two pairs $\{S_i, S_j\}$ of Σ in respective configurations s_i and s_j are described by an element D_{ij} of a Lie subgroup $L(s_i, s_j)$ of $\mathbb{D}(\mathcal{E})$. Such an assumption describes in fact a large part of usual links in rigid body systems except for example pure unilateral contact links.

With these tools, we can then define an observational frame \mathcal{F} and the way to pass from any frame \mathcal{F}_1 to any other frame \mathcal{F}_2 . We may also define the class \mathcal{Gal} of Galilean or inertial frames. The principle of mechanics (time denoted t is supposed to be absolute) consists then of two parts: the first one is a statement valid for any observational frame, the second one deals with the class \mathcal{Gal} .

1. If S is a rigid body with \mathbb{S}_S as configuration space, if \mathcal{F} is any frame of observation of the motion $t \mapsto s(t)$ of S , if the external actions observed with respect to \mathcal{F} are described by an element $F_{\text{ext}}(t) \in \mathcal{D}(\mathcal{E})$ and if $F_{\text{in}}(t)$ denotes the element of $\mathcal{D}(\mathcal{E})$ describing the inertial actions on S due to its motion with respect to \mathcal{F} , then for all t :

$$F_{\text{in}}(t) + F_{\text{ext}}(t) = 0. \tag{2}$$

2. If $\mathcal{F} \in \mathcal{Gal}$ is any inertial frame for the observation of the motion $t \mapsto s(t)$ of S then

$$F_{\text{in}}(t) = -\frac{d}{dt}(H_{s(t)}(V(t))) = -H_{s(t)}(\dot{V}(t)) - [V(t), H_{s(t)}(V(t))] \tag{3}$$

where $\theta(\dot{s}(t)) = V(t)$ for all t .

2.2 Three remarkable results

To illustrate the efficacy of such an intrinsic approach, we now present three results that have been obtained thanks to this Lie group geometric approach.

2.2.1 Objectivity of inertial actions

The first result we have chosen to present highlights the fundamental meaning of the objectivity of inertial forces and the associated properties of the “mass” of a body in order that the associated inertial force is an objective quantity. The main references are (Chevallier 2004) and more recently (Chevallier and Lerbet 2018).

Let \mathcal{F}_1 and \mathcal{F}_2 be two observation frames of the motion of a rigid body S . They induce two maps $t \mapsto s_i(t)$ in \mathbb{S}_S for $i = 1, 2$ so that the properties of the group action on \mathbb{S}_S leads to a C^∞ map $t \in \mathbb{R} \mapsto A_{12}(t) \in \mathbb{D}(\mathcal{E})$ with $s_1(t) = A_{12}(t) \bullet s_2(t)$ for all t . We then define objectivity in the framework of rigid body dynamics.

Definition 1 *A mechanical quantity attached to S and described by elements $X_i : t \mapsto X_i(t) \in \mathcal{D}(\mathcal{E})$ when the motion of S is observed with respect to the frames $\mathcal{F}_i, i = 1, 2$, is said to be objective or frame-indifferent if they are related by $X_1(t) = Ad(A_{12}(t))X_2(t)$ for all t .*

Definition 1 is a mathematical translation of philosophical ideas presented by Noll (1974) and other great authors but mainly for continuum mechanics and surprisingly (almost) never investigated for inertial actions. For example, the velocity field V of S is not an objective quantity: in fact a

calculation shows that $V_2(t) = Ad(A_{12}(t))(V_1(t) + U_{12}(t))$ with $U_{12}(t) = \theta_\ell(\dot{A}_{12}(t))$ where θ_ℓ is the left-Maurer-Cartan form of the Lie group $\mathbb{D}(\mathcal{E})$. In contrast, any force F acting on S is (must be!) an objective quantity. We stress that objectivity is a fundamental property in mechanics and is related to any change of frames and not only of inertial frames. We then question the objectivity of the inertial forces. The question is: are the forces defined in Galilean frames by the Expression (3) objective quantities?

Keep in mind that until now the kinetics of a rigid body S is not defined from a model of the body (distribution of particles for example that necessitates to accept the existence of an hypothetical object without volume but with mass) but through a field $s \in \mathbb{S}_S \mapsto H_s \in \mathcal{L}(\mathcal{D}(\mathcal{E}))$ that produces, from the velocity field $V(t)$ at time t , the kinetic momentum $H_{s(t)}(V(t))$ of the body S ($t \mapsto s(t)$ is the motion and $\theta(\dot{s}(t)) = V(t)$ is the velocity field of the body). The general properties of this field are:

1. for all $s \in \mathbb{S}_S$, H_s is a symmetric definite positive (for the Klein form $[\cdot | \cdot]$) operator of the Lie algebra (mathematical translation of the fundamental property of the kinetic energy),
2. the map $s \mapsto H_s$ is equivariant relatively to the left action \bullet : for all $s \in \mathbb{S}_S$ and $A \in \mathbb{D}(\mathcal{E})$, $H_{A \bullet s} = AdA \circ H_s \circ AdA^{-1}$ (mathematical translation of the fact that the “mass” of the rigid body rigidly follows the body during any motion).

We introduce the following field $s \mapsto C_s$ of bilinear equivariant maps C_s of $\mathcal{L}(\mathcal{D}(\mathcal{E}))$:

$$C_s(X, Y) = [X, H_s(Y)] + [Y, H_s(X)] + H_s([X, Y]), \quad \forall s \in \mathbb{S}_S, \quad \forall X, Y \in \mathcal{D}(\mathcal{E}). \quad (4)$$

The equivariant property of $s \mapsto C_s$ comes from the one of $s \mapsto H_s$ and the fact that the adjoint map is a Lie algebra representation. It reads $C_{A \bullet s}(X, Y) = C_s(AdA^{-1}(X), AdA^{-1}(Y))$ for all $s \in \mathbb{S}_S$, $A \in \mathbb{D}(\mathcal{E})$ and $X, Y \in \mathcal{D}(\mathcal{E})$.

The fundamental result is as follows.

Theorem 1 *Inertial forces acting on a rigid body S are objective if and only if the field of inertia operators $s \in \mathbb{S}_S \mapsto H_s \in \mathcal{L}(\mathcal{D}(\mathcal{E}))$ is such that:*

$$\forall X \in \mathfrak{I}(\mathcal{E}), \quad Y \in \mathcal{D}(\mathcal{E}), \quad s \in \mathbb{S}_S, \quad C_s(X, Y) = 0 \quad (5)$$

and by the equivariance property, it is sufficient that this property holds for only one configuration $r \in \mathbb{S}_S$.

A simple calculation then shows that, when the field of inertia operators $s \in \mathbb{S}_S \mapsto H_s \in \mathcal{L}(\mathcal{D}(\mathcal{E}))$ comes from a pre-defined mass distribution (for example a pre-defined volume density of mass), the objectivity condition (5) then holds. Obviously, such an analysis is in fact absolutely impossible without using such an intrinsic language allowing to focus only on the essential objects and to highlight the fundamental rule of the operators C_s through statement (5).

2.2.2 Full description of singularities of mechanisms

This second illustration of the power of the intrinsic geometric Lie group approach deals with pure kinematics and more exactly with mechanism theory. Roughly speaking, a mechanism is a rigid body system such that the graph of the links of the system is a closed chain. For open chains and for a large class of links, the configuration space of the system is (up to a rigid displacement) a differentiable manifold. Its dimension follows directly from the nature of each link of the system and especially from the degree of freedom of each joint. For closed chains, any formula for computing the mobility of the closed chain has always exceptions corresponding to “paradoxical” mechanisms. Moreover, even non paradoxical mechanisms may present singular configurations whose description escape the classical approach. In fact, the geometric approach mixing Lie group calculations and theory of singularities of maps (and the so-called stratification concept) allowed to definitively solve this long-standing issue. In the present paragraph, we not only present the ideas of this solution method but also explain why the usual approach is not successful. All considerations are local even if global questions can also be tackled. Many references about mechanism theory are available but for this short presentation, references are (Lerbet 1987), (Lerbet 1998) and again (Chevallier and Lerbet 2018) for a synthetic view.

It is not the objective here to present in details the most general theory. Assumptions will be made to simplify the understanding and are in fact not restrictive, see (Chevallier and Lerbet 2018). We will suppose that each mechanical joint L_k , $k = 1, \dots, n$ of n such joints, has only one degree-of-freedom. The relative motion allowed by this joint is then described by a so-called one-parameter subgroup of $\mathbb{D}(\mathcal{E})$ defined by an element ξ_k of the Lie algebra $\mathcal{D}(\mathcal{E})$. The configuration space \mathbb{V} of the mechanism made by the family L_k is then (locally in a neighborhood U of $(0, \dots, 0)$ in \mathbb{R}^n) given by the implicit equation:

$$f(q) = \exp(q_1 \xi_1) \dots \exp(q_n \xi_n) = e = id_{\mathbb{D}(\mathcal{E})} \quad (6)$$

where $e = id_{\mathbb{D}(\mathcal{E})}$ is the neutral element or the identity of the group $\mathbb{D}(\mathcal{E})$ and \exp is the usual exponential map of a Lie group. $f : \mathbb{R}^n \rightarrow \mathbb{D}(\mathcal{E})$ is also called the closure map or closure function of the mechanism. For $q \in \mathbb{R}^n$ and $k \in \{1, \dots, n\}$, we write $A_k(q_k) = \exp q_k \xi_k$ and $\mu_k(q) = \text{Ad}(A_1(q_1) \dots A_k(q_k))(\xi_k)$.

Regular mechanisms As mentioned in the introduction, the power of the method lies in the fact that all the derivatives are computed explicitly at this level and do not necessitate to introduce bases where vectors ξ_k of the Lie algebra become column vectors in \mathbb{R}^6 . The first derivative of f is given by:

$$Df(q) : x = (x_1 \dots, x_n) \mapsto \delta_{f(q)}^T \sum_{k=1}^n x_k \mu_k(q) = \delta_{f(q)}^T f_1(q) \quad (7)$$

where $\delta_A : B \in \mathbb{D}(\mathcal{E}) \mapsto BA \in \mathbb{D}(\mathcal{E})$ is the right multiplication on the Lie group $\mathbb{D}(\mathcal{E})$, $f_1 : \mathbb{R}^n \rightarrow \mathcal{L}(\mathbb{R}^n, \mathcal{D}(\mathcal{E}))$ is the right first differential of f . Rank analysis for Df may then be reported to f_1 . The mechanism is regular at q_0 if and only if the rank $r(q_0)$ of $f_1(q_0)$ is maximal and equal to $r_m = \max_{q \in \mathbb{R}^n} \text{rank}(f_1(q))$. In this regular case, classical results of differential geometry hold: \mathbb{V} is a differential manifold at q_0 with $\dim \mathbb{V} = n - r_m$, its tangent space at q_0 is $T_{q_0} \mathbb{V} = \ker f_1(q) = \{x = (x_1 \dots, x_n) \mid \sum_{k=1}^n x_k \mu_k(q_0) = 0\}$ and we can provide the Taylor expansion of a family of r_m dependent variables as functions of the other $n - r_m$ independent variables.

Weak singular mechanism Suppose now that $\text{rank}(f_1(q_0)) = r < r_m$ (often, $r = r_m - 1$) and let us define $\Sigma_r = \{q \in \mathbb{R}^n \mid \text{rank} f_1(q) = r\}$, called the r -class of f and $E^r = \{u \in \mathcal{L}(\mathbb{R}^n, \mathcal{D}(\mathcal{E})) \mid \text{rank} u = r\}$ (In fact, we must replace $\mathcal{D}(\mathcal{E})$ by the Lie algebra \mathfrak{Q} generated by ξ_1, \dots, ξ_n which can be strictly included in $\mathcal{D}(\mathcal{E})$ as, for example, in planar mechanisms). Results of differential geometry allow to equip $\Sigma_r(f) = f_1^{-1}(E^r)$ with a submanifold structure when a so-called transversality condition holds, that allows to transfer the submanifold structure from E^r to the class $\Sigma_r(f)$. This condition involves the derivative f_2 of f_1 .

Proposition 2 *The derivative of f_1 is given by*

$$f_2(q) = Df_1(q) : x \in \mathbb{R}^n \mapsto f_2(q)(x, y) = \sum_{k=1}^n \sum_{\ell < k} x_\ell y_k [\mu_\ell(q), \mu_k(q)]. \quad (8)$$

This map has very nice properties of symmetry and skew-symmetry, see (Chevallier and Lerbet 2018). The transversality condition is technical: it means that $\{y \mapsto f_2(x, y) \mid x \in \mathbb{R}^n\}$ is sufficiently large to contain all the linear maps from $K_q = \ker f_1(q)$ to a supplementary space of $F_q = \text{span}(\mu_1(q), \dots, \mu_n(q))$ in the Lie algebra \mathfrak{Q} and can be brought back to a rank calculation of a matrix. When this transversality condition holds, $\Sigma_r(f)$ is a submanifold whose tangent space is given by

$$T_{q_0} \Sigma_r(f) = \{x \in \mathbb{R}^n \mid f_2(q_0)(x, y) \in F_{q_0}, \forall y \in K_{q_0}\}. \quad (9)$$

If \mathbb{V} is locally included in $\Sigma_r(f)$ at q_0 , then \mathbb{V} is locally a submanifold (q_0 is then called a weak singular configuration) and the tangent space at q_0 is given by

$$T_{q_0} \mathbb{V} = \{x \in K_{q_0} \mid f_2(q_0)(x, x) = \sum_{k=1}^n \sum_{\ell < k} x_\ell x_k [\mu_\ell(q_0), \mu_k(q_0)] \in F_{q_0}\}. \quad (10)$$

The condition is quadratic and not linear and does not define *a priori* a vector space but only a cone. In this case, we can prove that the cone is a vector subspace.

Remark 1 In many books of robotics, we can find elements of calculation that start by the map f (the geometric model), its first derivative (the Jacobian $J(f)$ of f namely the so-called kinematic model) and the dynamic model that necessitate the second derivative of f or the derivative of $J(f)$. Unfortunately, these expressions are completely general and purely formal as in Equation (1) and do not contain the necessary explicit information that allows to concretely produce significant results. In the same vein, kinematicians from the XIXth and XXth centuries had in hand a relation of the type

$$\sum_{k=1}^n \dot{x}_k \xi_k = 0 \tag{11}$$

describing the first order condition of mobility. It necessitates only the vector space structure of $\mathcal{D}(\mathcal{E})$ (screw theory). Regarding the second order condition shown in Equation (10) involving Lie brackets, the Lie algebra structure is unavoidable to discover the full conditions of mobility.

Strong singular mechanisms It is also possible that the transversality condition fails or more often that \mathbb{V} is not locally included in $\Sigma_r(f)$ at q_0 . It is the situation when \mathbb{V} is no longer a differential manifold at q_0 . q_0 is then a singular point according to the meaning of singular points for algebraic or analytic varieties. In this case, there is no tangent vector space at q_0 . A beautiful example of such a situation in mechanism theory is when we open and/or close a music stand. When it is open, we are on one branch of the algebraic or analytic variety and when it is closed, we are on the other branch: the intersection point of the two branches is a such strong singular configuration that allows in fact to concretely close and open the music stand. It means that singular configurations are not always bad configurations that should be avoided.

The quadratic relation in Equation (10) no longer defines a vector space but only a cone $C_{2,q_0}\mathbb{V}$. Calculating then the successive derivatives of f_3, f_4, \dots of f_1 involving multi Lie brackets allows us to define a decreasing sequence $(C_{k,q_0}\mathbb{V})_k$ of cones and thus to compute explicitly the intersection of all these cones denoted $C_{q_0}\mathbb{V}$ which is the tangent cone at the singularity q_0 . The decreasing sequence is necessarily stationary for some integer n_0 that depends on the mechanism. Very often, $n_0 = 2$ and $C_{q_0}\mathbb{V} = C_{2,q_0}\mathbb{V}$ like for the music stand. Explicit expressions of the successive derivative and the corresponding equations of the cones $C_{k,q_0}\mathbb{V}$ can be found partially in (Lerbet 1987) and completely in (Lerbet 1998) and in (Chevallier and Lerbet 2018) where singular open chains are also tackled.

2.3 Beam theory

2.3.1 General partial differential equation

The third example chosen to illustrate applications of Lie group calculations in mechanics is beam theory, namely one-dimensional continuous mechanical system theory; we call it also curvilinear system theory. This subject is among the most extensively studied in mechanics, with foundational concepts dating back to the works of Leonardo da Vinci. As for any structural mechanics point of view, the idea is to use some privileged direction or dimension in the physical system Σ to propose an alternative description to the 3-dimensional usual one. For beam theory, it means that one dimension is privileged that will be described by a parameter σ belonging to an interval $[0, \ell_0]$ of \mathbb{R} . The choice of this parameter means that we choose a reference configuration r_Σ for Σ . Many models suppose that each material section S_σ attached to the value σ along the privileged direction remains rigid during the possible transformation of Σ . We adopt this model and discuss this hypothesis in Section 2.4. For this application to beam theory, references are (Lerbet 1989; Lerbet 2003) but the main and complete one is (Hariz et al. 2024).

The configuration space \mathbb{S}_Σ of Σ is then equal to $\prod_{\sigma \in [0, \ell_0]} \mathbb{S}_\sigma$. Since the configuration $r_\Sigma = (r(\sigma))_{\sigma \in [0, \ell_0]}$ with $r(\sigma) \in \mathbb{S}_\sigma$ for each $\sigma \in [0, \ell_0]$ is given, any motion $t \in \mathbb{R} \mapsto (s_\sigma(t))_{\sigma \in [0, \ell_0]} \in \mathbb{S}_\Sigma$ is equivalent to a map $(\sigma, t) \in [0, \ell_0] \times \mathbb{R} \mapsto D(\sigma, t) \in \mathbb{D}$ such that $s_\sigma(t) = D(\sigma, t) \bullet r(\sigma)$ for any $(\sigma, t) \in [0, \ell_0] \times \mathbb{R}$. Such a curvilinear system is then a general Cosserat 1D system.

Applying the mechanic principles allows us to obtain the single partial differential equation in the Lie algebra $\mathcal{D}(\mathcal{E})$ governing the dynamic evolution. This unique equation is equivalent to a set of 6 scalar partial differential equations for such a curvilinear system. Again, these completely general equations cannot be obtained by hand without any additional hypothesis. This intrinsic dynamic equation reads in a pure Lagrangian form:

$$\rho_0 \mathcal{T}_0(\sigma, t) + \frac{\partial \Theta_0}{\partial \sigma}(\sigma, t) + [\mathbf{e}_0(\sigma, t), \Theta_0(\sigma, t)] = \rho_0 \left(H_0 \frac{\partial V_0(\sigma, t)}{\partial t} + [V_0(\sigma, t), H_0(V_0(\sigma, t))] \right) \quad (12)$$

where $V_0 = \theta_t(\frac{\partial D}{\partial t})$ and $\mathbf{e}_0 = \theta_t(\frac{\partial D}{\partial \sigma})$ are functions with values in the Lie algebra $\mathcal{D}(\mathcal{E})$ and describes velocity and deformation fields of Σ , \mathcal{T}_0 is the mass density of external actions, Θ_0 , the field of internal actions and H_0 , the mass density of inertia operators. These quantities are purely Lagrangian and refer to r_Σ . This reference configuration is any and can be freely chosen according to the concrete cases. Constitutive laws provide additional relations between Θ_0 and \mathbf{e}_0 . The usual boundaries conditions involve Θ_0 , \mathbf{e}_0 and/or D (D corresponds to Dirichlet conditions) for $\sigma = 0$ or ℓ_0 so that we get a well-founded partial differential problem in $\mathcal{D}(\mathcal{E})$.

A lot of comments could be made on this equation and its remarkable structure. On both sides of the equation (apart the external actions), we see a Lie bracket appear, one for the internal actions and one for the inertia terms. They comes from the derivative of the adjoint map of the Lie group and are present on any form of the mechanic equations where the group is explicitly parametrizing the motion of the mechanical system, see (Poincaré 1901; Arnold 1966; Chevallier 1984). We now illustrate how further developments can be led while remaining coordinate-free.

2.3.2 Free-Vibrations of beams

In a first time we show how a complete and general coordinate-free linearization around any equilibrium is possible. Suppose then first that there is an equilibrium position of Σ when it is subjected to a distributed load $\mathcal{T}_{e,0}(\sigma)$. Equilibrium is then defined by the two fields $\sigma \mapsto D_{e,0}(\sigma)$ (leading to the field $\sigma \mapsto \mathbf{e}_{e,0}(\sigma)$) and $\sigma \mapsto \Theta_{e,0}(\sigma)$. The equilibrium equation reads

$$\rho_0 \mathcal{T}_{e,0}(\sigma) + \frac{\partial \Theta_{e,0}}{\partial \sigma}(\sigma) + [\mathbf{e}_{e,0}(\sigma), \Theta_{e,0}(\sigma)] = 0 \quad (13)$$

As usual we suppose that the displacement field $(\sigma, t) \mapsto D(\sigma, t)$ is the superposition of the equilibrium displacement field $\sigma \mapsto D_{e,0}(\sigma)$ and a time depending field. Since the last field is supposed to be small we express it thanks to the exponential map of the Lie group so that $D(\sigma, t) = \exp X(\sigma, t) \circ D_{e,0}(\sigma)$ where the kinematic unknown is now a field $(\sigma, t) \mapsto X(\sigma, t)$ with values in the Lie algebra. The power of the Lie group approach arises by leading to the first order expression (12) with respect to $X(\sigma, t)$ and its derivatives. By duality, we also suppose that the magnitude of $\chi_0(\sigma, t) = \Theta(\sigma, t) - \Theta_{e,0}(\sigma)$ is of same order than the one of $X(\sigma, t)$. Using Equation (13) in a first order approximation of Equation (12) leads to the equation

$$\rho_0 H_0 \left(\text{Ad} D_0^{-1}(\sigma) \frac{\partial^2 X(\sigma, t)}{\partial t^2} \right) = \frac{\partial \chi_0}{\partial \sigma}(\sigma, t) + \left[\text{Ad} D_0^{-1}(\sigma) \left(\frac{\partial X(\sigma, t)}{\partial \sigma} \right), \Theta_{e,0}(\sigma) \right] + [\mathbf{e}_{e,0}(\sigma), \chi_0(\sigma, t)] \quad (14)$$

which is the most general vibration equation of a Cosserat beam (no constitutive law has yet been chosen). The two unknown fields are $(\sigma, t) \mapsto X(\sigma, t)$ and $(\sigma, t) \mapsto \chi_0(\sigma, t)$. Obviously a constitutive law linking these two fields must be added to have a closed formulation. To lead to a usual form with only the kinematic field $(\sigma, t) \mapsto X(\sigma, t)$ as unknown, we now suppose that vibrations are elastic.

2.3.3 Free-Vibrations of elastic beams

Linear elastic vibrations means here that there is an operator E_0 (we suppose the material homogeneous so that E_0 does not depend on σ) such that $\chi_0(\sigma, t) = E_0(\text{Ad} D_{e,0}^{-1}(\sigma) \frac{\partial X(\sigma, t)}{\partial \sigma})$. Using the equilibrium configuration as a new reference configuration consists in applying $\text{Ad} D_{e,0}(\sigma)$ to Equation (14) by the left. The final expression is then

$$\begin{aligned} \rho_0 \tilde{H}_0 \left(\frac{\partial^2 X(\sigma, t)}{\partial t^2} \right) &= \tilde{E}_0 \left(\frac{\partial^2 X(\sigma, t)}{\partial \sigma^2} \right) - \tilde{E}_0 \left(\left[\tilde{\mathbf{e}}_0(\sigma), \frac{\partial X(\sigma, t)}{\partial \sigma} \right] \right) \\ &+ \left[\frac{\partial X(\sigma, t)}{\partial \sigma}, \tilde{\Theta}_{e,0}(\sigma) \right] + \left[\tilde{\mathbf{e}}_0(\sigma), \tilde{E}_0 \left(\frac{\partial X(S_0, t)}{\partial S_0} \right) \right] \end{aligned} \quad (15)$$

with $\tilde{H}_0 = \tilde{H}_0(\sigma) = \text{Ad}D_{e,0}(\sigma) \circ H_0 \circ \text{Ad}D_{e,0}^{-1}(\sigma)$, $\tilde{E}_0 = \tilde{E}_0(\sigma) = \text{Ad}D_{e,0}(\sigma) \circ E_0 \circ \text{Ad}D_{e,0}^{-1}(\sigma)$ and $\tilde{\mathbf{e}}_0(\sigma) = \text{Ad}D_{e,0}(\sigma)(\mathbf{e}_0(\sigma))$ are the corresponding quantities pushed forward from the reference configuration to the equilibrium one. One recognizes the right-hand side term of the equivariant property of inertia operators.

This equation explicitly contains all the coupling terms between the vibrations and the equilibrium configuration and supposes linear elasticity only for the small motions. The calculations of the equilibrium may involve non linear elasticity or even inelastic transformations.

2.3.4 Variational formulation for spectral analysis

The last illustration of Lie group calculations deals with spectral analysis and variational formulation. Looking for time harmonic solution $X(\sigma, t) = \sin \omega t U(\sigma)$ leads to find a function $\sigma \mapsto U(\sigma)$ with some convenient properties of regularity (derivative could be understood in the sense of distributions) and verifying some boundary conditions we do not now specify. The map $\sigma \mapsto U(\sigma)$ is then solution of the differential equation

$$\tilde{E}_0(U''(\sigma) - [\tilde{\mathbf{e}}_0(\sigma), U'(\sigma)]) + [U'(\sigma), \tilde{\Theta}_{e,0}(\sigma)] + [\tilde{\mathbf{e}}_0(\sigma), \tilde{E}_0(U'(\sigma))] + \rho_0 \omega^2 \tilde{H}_0(U(\sigma)) = 0 \quad (16)$$

Using the Klein form $[\cdot | \cdot]$ allows us to define a scalar product on the space of functions defined on $[0, \ell_0]$ (axioms of scalar product can be checked due to properties of inertia operators of rigid bodies) denoted $\llbracket \cdot | \cdot \rrbracket$ and associated with the kinetic energy of Σ by

$$\llbracket U | V \rrbracket = \int_0^{\ell_0} \rho_0 [\tilde{H}_0(U(\sigma)) | V(\sigma)] d\sigma \quad (17)$$

The spectral analysis is then brought back to the analysis of the bilinear form $(U, V) \mapsto \mathfrak{E}(U, V)$ where

$$\mathfrak{E}(U, V) = \int_0^{\ell_0} [\tilde{E}_0(U''(\sigma)) - \tilde{E}_0([\tilde{\mathbf{e}}_0(\sigma), U'(\sigma)]) + [U'(\sigma), \tilde{\Theta}_{e,0}(\sigma)] + [\tilde{\mathbf{e}}_0(\sigma), \tilde{E}_0(U'(\sigma))] | V(\sigma)] d\sigma \quad (18)$$

and more precisely to the operator A such that $\mathfrak{E}(U, V) = \llbracket A(U) | V \rrbracket$. The symmetry of \mathfrak{E} is especially important in stability analysis which can be further pursued when boundary conditions are specified.

2.4 Some research avenues

Numerous authors use Lie group approaches (often partially by fixing a point in the space and describing then the group $\mathbb{D}(\mathcal{E})$ by a semi-direct product of translations and rotations about this point) in robotics. As developed in Section 2.3 this approach has also been extended to investigate continuous systems like Cosserat beams providing a unique partial differential equation in the Lie algebra $\mathcal{D}(\mathcal{E})$ without any coordinate system and allowing to make all the derivative calculations for example for a vibration analysis in a complete intrinsic approach. In the same context, a similar approach has also been used for the control of an artificial model of eel locomotion by F. Boyer (see for example (Boyer et al. 2012)) leading finally to very performing prototypes. Strangely, the control theory does not systematically use this approach and investigates the controllability of each mechanical system from the scalar equations derived from Lagrange general equations leading to equations like Equation (1) that are purely formal and do not translate the specifics of each system. As already mentioned, it is quasi impossible to generate these specific equations by hand for a little bit complicated mechanical system and it is the reason why tools of control theory are often used for mechanical systems with a very small number of degrees of freedom. We think that it can be a large field of investigations to generalize the coupling of the control theory tools with the intrinsic approach by Lie group theory.

Another very important aspect for a large number of mechanical systems targeted by so-called mobile robotics is the intuitive discrimination of “global rigid motion” of the system and “deformation” of the system. For example for a rocket, the global rigid motion is controlled and the deformation (often vibrations) is passively subjected whereas for the artificial model of the eel mentioned above, the control is made on the “internal degrees of freedom” (namely

the deformation) in order to produce automatically a “global rigid motion” (viewed here as the motion of the head of the eel). A third situation between these two extreme dual situations is for example illustrated by artificial satellites where the global motion (around the earth for example) and the deformation are both controlled. In fact, this aspect of the dynamics falls under the field of the dynamics in a principal fiber bundle, see for example (Chevallier 2011; Bloch 2015) where the Lagrangian or Hamiltonian points of view are preferred. Roughly speaking, a principal fiber bundle is a structure equipping a set on which a Lie group acts non transitively. It is the general case of a configuration space of a rigid-body system. The quotient space by this action (technical hypotheses must be considered) is the space of shapes of the system and called the base of the fiber bundle and each fiber over the base is the set of configurations frozen with the corresponding shape, the motion along this fiber over this shape being rigid. The concept of connection in such a fiber space with their associated concepts of curvature and torsion is then central to geometrically describe the dynamics in such a space. As noted above and according to the targeted system, the control can be done along the fibers, on the base or in a coupled way. This geometric view can be a deep help for control theory and we are convinced that a larger use of this geometric approach should be known, shared and used in the community of mechanics since it decouples as far as possible the two sides of the general motion. An optimal numerical approach should also then start from these intrinsic equation set. Section 3 also considers this structure in a complete different framework.

Regarding the intrinsic beam theory, we could point out another interesting line of research. It should consist in considering the group acting on each section of the curvilinear system to be the full affine group $\mathcal{G}a(\mathcal{E})$ and not the displacement group $\mathbb{D}(\mathcal{E})$ which is a subgroup of $\mathcal{G}a(\mathcal{E})$. A general investigation for a single body can be found in (Burov and Chevallier 2008). It should allow to take into account a part of the deformations of the sections which necessarily occur in the real life.

3 Granular mechanics

The second topic that we now consider comes from recent research regarding granular mechanics. It explains the section title even though the general context of these developments is the quasi-static evolution of any finite degree-of-freedom system. In a certain sense, it is a framework opposite to the one dealt in Section 2. In this context, the inertia of the elementary objects (the grains) will first be neglected (incremental quasi-static evolution) and we will essentially focus on the incremental law of the evolution of the system. It could be surprising that the most common numerical method to deal with such systems, that is the Discrete Element Method, mainly involves Newton’s laws for each grain. It is in fact a way to deal with potential discontinuities like shear band (Cundall and Strack 1979) and continual re-arrangement of the grain system. The following investigations build upon previous research regarding stability in geomechanics, with a particular focus on granular mechanics. The initial and main purpose of these investigations was to reconcile an instability criterion dealing with the tangent stiffness matrix K (the usual Euler criterion $\det K = 0$) and the instability criterion involving the symmetric part K_s of K (the second order work criterion or Hill’s criterion $\det K_s = 0$). A complete view of this question and its solution can be found in (Lerbet et al. 2020a; Lerbet et al. 2020b). The solution uses the new concept of Kinematic Structural Stability (KiSS) and its dual object of geometric degree of non conservativity (GDNC) which are really significant for nonconservative systems leading to non symmetric stiffness matrices (hypo-elastic systems, non-associate plastic systems, follower external actions,...).

A large part of the solution is in fact algebraic. In the geometric language, it means that the calculations are conducted “over a point”. Dealing with a complete evolution means to pass from the punctual point-of-view to (at least!) a local one. To remain intrinsic, this passage must not depend on a choice of coordinate system. The way to pass from the punctual to the local or even global viewpoint naturally leads to the language of vector bundles. An additional difficulty, already seen in Section 2.4, is the necessity to distinguish, in the framework of discrete systems, the deformations (change of shape) and the displacements, namely the principal bundle structure of the configuration space. Main references for this part are (Lerbet et al. 2018; Lerbet et al. 2023).

While the Polish school is highly active in the field of geometric mechanics—addressing these frameworks and more complex structures such as Dirac structures and groupoids—our objective is more modest. Similar to the Lie group approach, our goal is to demonstrate how a suitable geometric language can clarify outdated concepts and provide a more efficient framework for performing and understanding calculations.

3.1 A geometric approach

The proposed geometric approach borrows concepts from Lagrange's formalism. For any differential map $f : M \rightarrow N$ between two manifolds M and N , $f^T : TM \rightarrow TN$ is its tangent map.

1. The configuration set of the system Σ is supposed to be a finite dimension manifold \mathbb{M} . The natural non-transitive action of a Lie subgroup \mathbb{G} (depending on each system Σ) of the Lie group $\mathbb{D}(\mathcal{E})$ on \mathbb{M} (with some technical assumptions) allows to define a principal bundle structure $(\mathbb{M}, \mathbb{G}, \mathbb{B}, \omega)$ on \mathbb{M} . \mathbb{B} is then the space of shapes of Σ . If $\epsilon = \omega(m)$, the fiber $\omega^{-1}\{\epsilon\}$ is a submanifold diffeomorphic to the Lie group \mathbb{G} . If two configurations m and m' are on the same fiber (namely if $\omega(m) = \omega(m')$) there is a unique rigid displacement $g_{mm'} \in \mathbb{G}$ such that $m' = g_{mm'} \bullet m$.
2. $T\mathbb{M} = \bigcup_{m \in \mathbb{M}} T_m\mathbb{M}$ is the (total space of the) tangent bundle of \mathbb{M} . A vector $\delta m \in T_m\mathbb{M}$ describes any infinitesimal displacement or a rate of displacement in the configuration m . This infinitesimal displacement is a rigid infinitesimal displacement if $\omega^T(\delta m) = 0$. A rigid infinitesimal displacement is denoted $\delta^r m$. The set of infinitesimal rigid displacement in the configuration m is the vector subspace $\omega^{T^{-1}}\{(\omega(m), 0)\} = T_m\omega^{-1}\{\omega(m)\}$ of $T_m\mathbb{M}$. The tangent bundle is denoted $\tau(\mathbb{M}) = (T\mathbb{M}, \mathbb{M}, \pi_{\mathbb{M}})$.
3. Generically the elements of \mathbb{B} describing the shapes are denoted ϵ . It means that there is a configuration m such that $\omega(m) = \epsilon$ and ϵ is then the shape of Σ in the configuration m . $T\mathbb{B}$ is the tangent bundle of \mathbb{B} . We stress that there is no canonical way to associate a vector δm of $T_m\mathbb{M}$ to a vector $\delta\epsilon$ of $T_\epsilon\mathbb{B}$ when $\epsilon = \omega(m)$ since if $\omega^T(\delta m) = \delta\epsilon$ then $\omega^T(\delta m + \delta^r m) = \delta\epsilon$ for any infinitesimal rigid displacement $\delta^r m$ at m . Elements $\delta\epsilon$ are called rates of pure deformation. The tangent bundle is denoted $\tau(\mathbb{B}) = (T\mathbb{B}, \mathbb{B}, \pi_{\mathbb{B}})$.
4. A punctual description of an external force ϕ acting on Σ in a configuration m is an element \mathbf{f} of $T_m^*\mathbb{M}$. A punctual description of an internal action ϕ_ℓ of Σ in the configuration m has two equivalent formulations. It is an element \mathbf{f}_ℓ of $T_m^*\mathbb{M}$ such that $\mathbf{f}_\ell(\delta^r m) = 0$ for any infinitesimal rigid displacement $\delta^r m$ in the configuration m or it is an element $\tilde{\mathbf{f}}_{\text{def}}$ of $T_\epsilon^*\mathbb{B}$ with $\omega(m) = \epsilon$. The relation between the two representations reads $\tilde{\mathbf{f}}_{\text{def}}(\delta\epsilon) = -\mathbf{f}_\ell(\delta m)$ if $\delta\epsilon = \omega^T(\delta m)$. It means that $\mathbf{f}_\ell = -\omega^*(\tilde{\mathbf{f}}_{\text{def}})$: \mathbf{f}_ℓ is the pull-back of $-\tilde{\mathbf{f}}_{\text{def}}$ by ω . The corresponding vector bundles are $\tau^*(\mathbb{M}) = (T^*\mathbb{M}, \mathbb{M}, p_{\mathbb{M}})$ with $T^*\mathbb{M} = \bigcup_{m \in \mathbb{M}} T_m^*\mathbb{M}$ the total space of the cotangent bundle $\tau^*(\mathbb{M})$ of \mathbb{M} and $\tau^*(\mathbb{B}) = (T^*\mathbb{B}, \mathbb{B}, p_{\mathbb{B}})$ with $T^*\mathbb{B} = \bigcup_{\epsilon \in \mathbb{B}} T_\epsilon^*\mathbb{B}$ the total space of the cotangent bundle $\tau^*(\mathbb{B})$ of \mathbb{B} . We often identify the vector bundle and its total space when no confusion is possible.
5. When we want to pass from a pure punctual point-of-view to a local one, the natural way in vector bundles like $\tau^*\mathbb{M}$ or $\tau^*\mathbb{B}$ is realized through the concept of cross-section or simply section. It is quite natural to suppose that all external actions may be described by such a section but we will distinguish fixed actions like gravity (section F_f of $\tau^*\mathbb{M}$) and control or loading actions that are the cause of the evolution of the system (section $F_c(\sigma)$ where σ is the load parameter). To simplify the presentation, we suppose that $F_c(0) = 0_{\tau^*\mathbb{M}}$ where $0_{\tau^*\mathbb{M}}$ is the nil section of the vector bundle $\tau^*\mathbb{M}$. We can add the sections of a vector bundle and $F_g(\sigma) = F_f + F_c(\sigma)$ is the section of the total external and given actions acting on Σ along the loading process. $\sigma \mapsto F_g(\sigma)$ is then a given function with values in the modulus $\Gamma(\tau^*\mathbb{M})$ of sections of $\tau^*\mathbb{M}$. The way to describe the local representation of internal actions is precisely the rule of the constitutive law of the system Σ . When forces are described by a section, we say that it is an integrable situation.

3.2 Geometric foundations of usual concepts

We now present some basic concepts for the integrable case in this coordinate-free framework.

3.2.1 Elasticity, hyperelasticity, weak hypoelasticity, strong hypoelasticity

We now translate the differentiable constitutive laws (elasticity, hyperelasticity, hypoelasticity) in this vector bundle language. The passage from integrable (or weak) to not integrable (or strong) hypoelasticity leads to pass from a first-order description to a second order description: this involves the so-called double vector bundle concept. We will not insist here on this more complicated structure.

1. Σ is elastic if there is a section \tilde{F}_{el} of $T^*\mathbb{B}$ such that (at least locally) for all $\epsilon \in \mathbb{B}$ the deformations forces $\tilde{\mathbf{f}}_{def} \in T_{\epsilon}^*\mathbb{B}$ in the system Σ in the shape state ϵ is the value of this section at ϵ : $\tilde{\mathbf{f}}_{def} = \tilde{F}_{el}(\epsilon)$.
2. If the 1-form \tilde{F}_{el} is exact, namely if there is a function U_{el} defined (at least locally) on \mathbb{B} such that $\tilde{F}_{el} = dU_{el}$, then Σ is called *hyperelastic*.
3. If the 1-form \tilde{F}_{el} is not closed, namely if the exterior derivative $d\tilde{F}_{el}$ is not nil, then Σ is called *weak-hypoelastic* or *integrable-hypoelastic*.
4. The most general description of hypoelasticity can no longer be done through a section of $T^*\mathbb{B}$ but through the infinitesimal aspect of sections, namely a connection. An incremental or strong hypoelastic or not integrable hypoelastic constitutive law is an (Ehresmann) connection \mathcal{H} on $T^*\mathbb{B}$ namely a differentiable field $\mathcal{H} : \tilde{\mathbf{f}}_{def} \in T^*\mathbb{B} \mapsto \mathcal{H}(\tilde{\mathbf{f}}_{def}) \subset T_{\tilde{\mathbf{f}}_{def}} T^*\mathbb{B}$ of so-called horizontal spaces of $TT^*\mathbb{B}$ (double vector bundle) such that

$$T_{\tilde{\mathbf{f}}_{def}} T^*\mathbb{B} = V(\tilde{\mathbf{f}}_{def}) \oplus \mathcal{H}(\tilde{\mathbf{f}}_{def}) \quad (19)$$

where $V(\tilde{\mathbf{f}}_{def})$ is the vertical space at $\tilde{\mathbf{f}}_{def}$ canonically identified (since it is a vector space) to $T_{\tilde{\mathbf{f}}_{def}}^* \mathbb{B}$ itself.

When the curvature of this connection vanishes it becomes integrable and we come back to weak hypoelasticity, the section F_{def} being the integral manifold of the differential system associated with the connection.

3.2.2 Equilibrium and T-stability of equilibrium

A configuration m_e is an equilibrium of Σ subjected to external given forces described by the section $F_g(\sigma)$ (σ is a load parameter as above) when the deformation forces $\tilde{\mathbf{f}}_{def} \in T_{\epsilon_e} T^*\mathbb{B}$ developed into Σ are such that

$$F_g(\sigma)(m_e) = \tilde{\omega}^*(\tilde{\mathbf{f}}_{def}) \quad (20)$$

with $\epsilon_e = \tilde{\omega}(m_e)$.

For an elastic system, in the general sense defined above, this equilibrium equation is equivalent to

$$F_t(\sigma)(m_e) = 0 \quad (21)$$

where $F_t = F_t(\sigma) = F_g(\sigma) - \tilde{\omega}^*(\tilde{F}_{el})$ is the section of $T^*\mathbb{M}$ of all the actions. Geometrically, we say that the section F_t intersects or cuts the nil section $0_{T^*\mathbb{M}}$ of the cotangent bundle at $m_e = m_e(\sigma)$.

To deal with stability of an equilibrium m_e , we must consider perturbations about m_e which geometrically means to investigate the nature of the intersection of the two sections F_t and $0_{T^*\mathbb{M}}$ at m_e . The concept of transversality (first introduced by René Thom) is the appropriate concept to do it: an equilibrium $m_e = m_e(\sigma)$ is said to be T-stable if these two sections $F_t(\sigma)$ and $0_{T^*\mathbb{M}}$ transversely intersect at $m_e(\sigma)$. Remark that this definition is purely geometric and is not related to any calculation of the stiffness matrix for example. Such an equilibrium will be called T-stable or regular. If $m_e(\sigma)$ is regular, thanks to properties of transversality, equilibria remain regular for small perturbations of the load parameter σ .

3.2.3 Vertical derivative and stiffness operator

In this paragraph, we deal with analytic features of the previous purely geometric aspects. As announced, in the last step of our path, we will come back to the more usual and accepted coordinate approach. The non-integrable framework is more complicated but it can be also treated as completely as the integrable case we now present, see (Lerbet et al. 2023).

The idea is to compute the derivative of Equation (21) since the surjective map $F_t(\sigma) : \mathbb{M} \rightarrow T^*\mathbb{M}$ is differentiable. Recall that, by definition of a cross-section, $p_{\mathbb{M}} \circ F_t(\sigma) = id_{\mathbb{M}}$. We obviously can consider the usual derivative $dF_t(\sigma)$ of $F_t(\sigma)$ but it is a map from $T\mathbb{M}$ to $TT^*\mathbb{M}$ that forgets the bundle structure of $T^*\mathbb{M}$. A connection on $T^*\mathbb{M}$ is a way to make derivatives of sections preserving the bundle structure but there is here, namely in the problem of incremental evolutions, no natural connection. The appropriate way here is to use the nil section $0_{T^*\mathbb{M}}$ as an horizontal space which leads to the intrinsic concept of vertical derivative $dF_t(\sigma)^{\text{ver}}(m_e)$ of $F_t(\sigma)$ at the equilibrium configuration m_e : $dF_t(\sigma)^{\text{ver}}(m_e)$ is a linear map from $T_{m_e}\mathbb{M}$ on $T_{(m_e,0)}V_{m_e}(T^*\mathbb{M}) \approx V_{m_e}(T^*\mathbb{M}) = T_{m_e}^*\mathbb{M}$ since the vertical space is a vector space. By definition, using the intrinsic geometric decomposition $T_{(m_e,0)}T^*\mathbb{M} = T_{m_e}\mathbb{M} \oplus T_{m_e}^*\mathbb{M}$ (valid *a priori* only at an equilibrium configuration), we have for all $\delta m \in T_{m_e}\mathbb{M}$:

$$dF_t(\sigma)(m_e(\sigma))(\delta m) = \delta m + dF_t(\sigma)^{\text{ver}}(m_e(\sigma))(\delta m) \quad (22)$$

The element $dF_t(\sigma)^{\text{ver}}(m_e) \in \mathcal{L}(T_{m_e}\mathbb{M}, T_{m_e}^*\mathbb{M})$ is called the stiffness operator at m_e . The transversality condition for T-stability of m_e is equivalent to the surjectivity of $dF_t(\sigma)^{\text{ver}}(m_e)$. For dimension reasons, the surjectivity of this map is in fact equivalent to its invertibility. Choosing a coordinate system $q = (q_1, \dots, q_n)$ on \mathbb{M} at m_e leads to a basis $(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n})$ of $T_{m_e}\mathbb{M}$ and to the dual basis (dq_1, \dots, dq_n) of $T_{m_e}^*\mathbb{M}$ canonically associated to the coordinate system $q = (q_1, \dots, q_n)$. The matrix of the stiffness operator $dF_t(\sigma)^{\text{ver}}(m_e)$ in these bases is the usual stiffness matrix and the transversality condition for T-stability of m_e is then equivalent to the invertibility of the stiffness matrix namely the usual Euler criterion of Structural Mechanics.

3.2.4 Intrinsic equations

Along a loading path, when the loading parameter is increasing from 0, the incremental evolution reads $\sigma \mapsto m_e(\sigma)$ and is ensured as long as $m_e(\sigma)$ is regular. The curve $\sigma \mapsto m_e(\sigma)$ is obtained incrementally in the following way. At step σ of the loading path, the equilibrium described by $m_e(\sigma)$ solution to Equation (21) and $F_t(\sigma)$ is disturbed by an infinitesimal increasing $\delta\sigma$. This leads to a perturbation $\frac{\partial F_c(\sigma)}{\partial \sigma} \delta\sigma$ of control external forces that are purely vertical. In order that the evolution remains on the nil section of equilibria, the full contribution of all the vertical components must vanish. The incremental equation then reads

$$\frac{\partial F_c(\sigma)}{\partial \sigma} \delta\sigma + dF_t(\sigma)^{\text{ver}}(m_e(\sigma))(\delta m) = 0 \quad (23)$$

and we can deduce the infinitesimal variation δm as long as $dF_t(\sigma)^{\text{ver}}(m_e)$ is invertible, namely as long as the T-stability condition holds:

$$m_e(\sigma) = \int_0^\sigma (dF_t(u)^{\text{ver}}(m_e(u)))^{-1} \left(\frac{\partial F_c(u)}{\partial u} \right) du. \quad (24)$$

Remark that the two vectors of the sum in Equation (23) do not have the same origin in $TT^*\mathbb{M}$. It means that their sum has no meaning in general but they can be added here only because they are vectors tangent to the same fiber $T_{m_e(\sigma)}^*\mathbb{M}$ (the vertical space over m_e which is a vector space). This geometric property of control forces is never underlined.

3.3 Some research avenues

In order not to be too much technical, we have dealt here only with the integrable case when all actions are described by cross-sections of the cotangent bundle. Then, we remark that the incremental Equation (23) finally reads in $T^*\mathbb{M}$. Dealing with the non integrable case imposes to write the incremental process fully in $TT^*\mathbb{M}$ dealing with a full second order process but leading however again to a coordinate-free formulation thanks to some appropriate projections in $TT^*\mathbb{M}$, see (Lerbet et al. 2023) for a complete presentation. The power of the approach is undoubtedly the most evident in this non integrable case.

Since all the operations leading to the equations are explicit and are finally reduced to algebraic equations and operations, it is possible to program all the process before dealing with

any particular system. The SageManifolds package developed at the LUX provides an appropriate good environment for the implementation. In (Lerbet et al. 2023), an example of a not integrable hypoelastic system illustrates the automatic process for building the incremental evolution equations. It shows how, once the geometric framework is well identified, this complex question is reduced even for non integrable systems, to only terms of building linear projections and of solving linear systems.

Regarding possible research developments in Mechanics, we may mention two ways. First, an extension to dynamics for hypoelastic discrete systems and an interesting presence of at least two geometric connections: the first one describing the constitutive law (the above Ehresman connection) and the Riemannian one linked to the kinetic energy of the system. It could be an original way to deal with instability in granular systems when, as it is observed in practice, they involve bursts of kinetic energy as they loose their stability. A second research avenue deals obviously with the direct extension of constitutive laws from hypoelasticity to plasticity. The deeply non differentiable nature of plasticity is a real challenge for any geometric method involving differential geometry.


4 Conclusion

In this paper, we aimed to briefly present two geometric tools fundamental to mechanics: Lie groups and vector bundles. We aimed to illustrate the power of these geometric approaches targeting various topics in mechanics: Nonlinear Dynamics, Robotics, Mechanisms Theory, Beam Theory, Granular materials, Constitutive laws, etc. Under the multiplicity and variety of applications, a constant message is present: the geometric language is not only a language for insiders but is in fact often the appropriate language for understanding the fundamental meaning of physical concepts. By aligning closely with our intuitive grasp of reality, this perspective often reveals the path toward solving long-standing open questions. However, what it is less often underlined is that this proximity with the “real world” does not distance these methods from the computational aspects and are in fact a good approach for a fruitful and optimal dialogue with computational mechanics. We began with a quote from René Thom and we will end with one from Jean-Marie Souriau: “Mathematics is like a pair of shoes for the hiker: it allows them to reach new summits and to observe the beauty of landscapes that would otherwise remain hidden”.

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